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On Oscillation and Nonoscillation of General Functional Differential Equations

S. R. GRACE AND B. S. LALLI*

Department of Mathematics, University of Saskatchewan, Saskatoon, Saskatchewan S7N 0W0, Canada

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Oscillation and nonoscillation criteria for the nth order nonlinear functional differential equation $L_n x(t) + f(t, x(t), x[g(t)]) = h(t)$ are established. Some illustrative examples are also included. © 1985 Academic Press, Inc.

1. INTRODUCTION

Consider the nth order nonlinear functional differential equation

$$L_n x(t) + f(t, x(t), x[g(t)]) = h(t), \tag{1}$$

where $L_0 x(t) = x(t)$, $L_k x(t) = a_k(t)(L_{k-1} x(t))^{\cdot}$, $k = 1, 2,..., n$, $a_n(t) = 1$, $(^{\cdot} = d/dt)$ $a_i, g, h\colon [t_0, \infty) \to R$ and $f\colon [t_0, \infty) \times R^2 \to R$ are continuous, $a_i(t) > 0$ for $i = 1, 2,..., n-1$ and $g(t) \to \infty$ as $t \to \infty$.

The domain $D(L_n)$ of L_n is defined to be the set of all functions $x\colon [t_0, \infty) \to R$ such that $L_j x(t)$, $0 \leqslant j \leqslant n$, exist and are continuous on $[t_0, \infty)$. By a solution of (1) we mean a function $x \in D(L_n)$ which satisfies (1) on $[t_0, \infty)$. A nontrivial solution of (1) is called oscillatory if the set of its zeros is unbounded and it is nonoscillatory otherwise.

In what follows we are primarily interested in obtaining conditions which ensure that certain type of solutions of (1) are nonoscillatory. Conditions which guarantee that oscillatory solutions of (1) with the property that $L_k x(t) \to 0$ as $t \to \infty$, $k = 0, 1,..., n-1$ are given.

The desired results are extablished in Section 2. These results generalize and improve some of the results of Graef, Kitamura, Kusano, Onose and Spikes [1] for the parlticular equation

$$(a(t)\, x^{\cdot}(t))^{\cdot} = f(t, x(t), x[g(t)]),$$

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and of Kusano and Onose [2, 3] for the equation

$$(a(t)\,x^{\cdot}(t))^{\cdot} + q(t) f(x[g(t)]) = h(t).$$

In Section 3 we apply these results to the functional equation

$$L_n x(t) + q(t) f(x[g(t)]) = h(t), \tag{2}$$

where q: $[t_0, \infty) \to R$, f: $R \to R$ are continuous, L_n, a_i, $i = 1, 2, \ldots, n-1$, g and h are as above, and obtain necessary and sufficient conditions so that all oscillatory solutions of (2) converge to zero as $t \to \infty$.

We assume that the function f satisfies an estimate of the form

$$|f(t, x, y)| \leqslant F(t, |x|, |y|), \tag{3}$$

where F: $[t_0, \infty) \times R_+^2 \to R_+$ is continuous and such that

$$F(t, u_1, v_1) \leqslant F(t, u_2, v_2) \qquad \text{for}$$
$$0 \leqslant u_1 \leqslant u_2 \qquad \text{and} \qquad 0 \leqslant v_1 \leqslant v_2.$$

Some of the results which follow are concerned with those solutions of (1) which satisfy growth estimates of the form

$$|x(t)| = O(m(t)) \qquad \text{as} \quad t \to \infty, \tag{4}$$

where m: $[t_0, \infty) \to R$ is continuous and positive.

2. Main Results

THEOREM 1. *Suppose that*

$$\int^{\infty} \frac{1}{a_1(s_1)} \int_{s_1}^{\infty} \frac{1}{a_2(s_2)} \int_{s_2}^{\infty} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{a_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s, cm(s), cm[g(s)])\, ds\, ds_{n-1} \cdots ds_1 < \infty \tag{5}$$

and

$$\int^{\infty} \frac{1}{a_1(s_1)} \int_{s_1}^{\infty} \frac{1}{a_2(s_2)} \int_{s_2}^{\infty} \cdots \int_{s_{n-1}}^{\infty} \frac{1}{a_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |h(s)|\, ds\, ds_{n-1} \cdots ds_1 < \infty, \tag{6}$$

for all $c > 0$. *If* $x(t)$ *is an oscillatory solution of* (1) *satisfying* (4), *then* $L_k x(t) \to 0$ *as* $t \to \infty$, $k = 0, 1, 2, \ldots, n-1$.

Proof. Let $x(t)$ be an oscillatory solution of (1) satisfying (4), then

$$|x(t)| \leqslant cm(t), \qquad |x[g(t)]| \leqslant cm[g(t)]$$
$$\text{for all } t \geqslant t_1 \geqslant t_0 \text{ and some } c > 0.$$

Since $x(t)$ is oscillatory, $L_k x(t)$ is oscillatory for $k = 1, 2, \ldots, n-1$. Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of consecutive zeros of $L_{n-1}x(t)$ and $\beta_m \in (t_m, t_{m+1})$ be such that

$$|L_{n-1}x(\beta_m)| = \max_{t_m \leqslant t \leqslant t_{m+1}} |L_{n-1}x(t)|.$$

Integrating (1) from t_m to β_m we obtain

$$L_{n-1}x(\beta_m) - L_{n-1}x(t_m) = -\int_{t_m}^{\beta_m} f(s, x(s), x[g(s)])\, ds + \int_{t_m}^{\beta_m} h(s)\, ds,$$

which gives

$$|L_{n-1}x(\beta_m)| \leqslant \int_{t_m}^{\beta_m} F(s, cm(s), cm[g(s)])\, ds + \int_{t_m}^{\beta_m} |h(s)|\, ds.$$

Now summing on m we have

$$\sum_{m=1}^{\infty} |L_{n-1}x(\beta_m)| \leqslant \int_{t_1}^{\infty} F(s, cm(s), cm[g(s)])\, ds + \int_{t_1}^{\infty} |h(s)|\, ds < \infty.$$

Consequently $\lim_{m \to \infty} L_{n-1}x(\beta_m) = 0$, which implies that $L_{n-1}x(t) \to 0$ as $t \to \infty$.

Integrating (1) from t to ∞ we have

$$L_{n-1}x(t) = \int_t^{\infty} f(s, x(s), x[g(s)])\, ds - \int_t^{\infty} h(s)\, ds. \tag{7}$$

Now we shall prove that $\lim_{t \to \infty} L_{n-2}x(t) = 0$. Let $\{t'_m\}_{m=1}^{\infty}$ be a sequence of consecutive zeros of $L_{n-2}x(t)$ and $\beta'_m \in (t'_m, t'_{m+1})$ be such that

$$|L_{n-2}x(\beta'_m)| = \max_{t'_m \leqslant t \leqslant t'_{m+1}} |L_{n-2}x(t)|.$$

Integrating (7) from t'_m to β'_m we obtain

$$L_{n-2}x(\beta'_m) = -\int_{t'_m}^{\beta'_m} \frac{1}{a_{n-1}(t)} \int_t^{\infty} f(s, x(s), x[g(s)])\, ds\, dt$$
$$- \int_{t'_m}^{\beta'_m} \frac{1}{a_{n-1}(t)} \int_t^{\infty} h(s)\, ds\, dt,$$

which implies

$$|L_{n-2}x(\beta'_m)| \leqslant \int_{t'_m}^{\beta'_m} \frac{1}{a_{n-1}(t)} \int_t^\infty F(s, cm(s), cm[g(s)])\, ds\, dt$$
$$+ \int_{t'_m}^{\beta'_m} \frac{1}{a_{n-1}(t)} \int_t^\infty |h(s)|\, ds\, dt.$$

Summing on m we get

$$\sum_{m=1}^\infty |L_{n-1}x(\beta'_m)| \leqslant \int_{t'_1}^\infty \frac{1}{a_{n-1}(t)} \int_t^\infty F(s, cm(s), cm[g(s)])\, ds$$
$$+ \int_{t'_1}^\infty \frac{1}{a_{n-1}(t)} \int_t^\infty |h(s)|\, ds\, dt < \infty.$$

Therefore $L_{n-2}x(t) \to 0$ as $t \to \infty$. Integrating (7) from t to ∞ we get

$$L_{n-2}x(t) = -\int_t^\infty \frac{1}{a_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty f(s, x(s), x[g(s)])\, ds\, ds_{n-1}$$
$$+ \int_t^\infty \frac{1}{a_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty h(s)\, ds\, ds_{n-1}.$$

Continuing this process we deduce that $L_k x(t) \to 0$ as $t \to \infty$, $k = 0, 1, ..., n-1$. This completes the proof of the theorem.

The following results are immediate consequence of Theorem 1; hence we omit the proofs.

THEOREM 2. *Assume that*

$$\int^\infty \frac{1}{a_i(s)}\, ds < \infty, \qquad i = 1, 2, ..., n-1, \tag{8}$$

$$\int^\infty F(s, cm(s), cm[g(s)])\, ds < \infty \tag{9}$$

$$\int^\infty |h(s)|\, ds < \infty, \tag{10}$$

for all $c > 0$. *If* $x(t)$ *is an oscillatory solution of* (1) *satisfying* (4), *then* $L_k x(t) \to 0$ *as* $t \to \infty$, $k = 0, 1, ..., n-1$.

COROLLARY 1. *Let in Theorem* 2 *the condition* (10) *be replaced by*

$$h(t)/F(t, cm(t), cm[g(t)]) \qquad \textit{is bounded,}$$
$$\textit{for all } c > 0, \quad t \geqslant t_0, \tag{10'}$$

then the conclusion of Theorem 2 *holds, provided that* $F(t, cm(t), cm[g(t)]) > 0$ *for* $t \geqslant t_0$.

THEOREM 3. *Assume that*

$$a_i'(t) \geqslant 0 \quad \text{or} \quad a_i(t) \geqslant \alpha > 0 \qquad \text{for} \quad i = 1, 2, ..., n-1, \tag{11}$$

$$\int^{\infty} s^{n-1} F(s, cm(s), cm[g(s)])\, ds < \infty, \tag{12}$$

$$\int^{\infty} s^{n-1} |h(s)|\, ds < \infty, \tag{13}$$

for all $c > 0$. *If* $x(t)$ *is an oscillatory solution of* (1) *satisfying* (4), *then* $L_k x(t) \to 0$ *as* $t \to \infty$, $k = 0, 1, ..., n-1$.

COROLLARY 2. *Let in Theorem* 3, *the condition* (13) *be replaced by* (10)′, *then the conclusion of Theorem* 3 *hold.*

COROLLARY 3. *In addition to* (8) *and* (10), *let condition* (5) (*or* (9) *or* (12)) *hold with* $m(t) = K$ *for every constant* K, *then every bounded solution* $x(t)$ *of* (1) *has the property that* $L_k x(t) \to 0$ *as* $t \to \infty$, $k = 0, 1, ..., n-1$.

Remarks. 1. If $n = 2$, $h = 0$, then Theorem 1 and Corollary 2 in [1] are included in our Theorem 1 and Corollary 3, respectively.

2. If $n = 2$ and $f(t, x, y) = q(t) f(x)$ or $q(t) f(y)$, then Theorem 1 and 4 in [2] are included in our Corollary 3.

For illustration we consider the following examples. In Examples 1, 2 and 3 we choose a_i so that $\int^{\infty} [1/a_i(s)]\, ds = \infty$, $i = 1, 2, ..., n-1$, and Example 4 concerns with the case when $\int^{\infty} [1/a_i(s)]\, ds < \infty$, $i = 1, 2, ..., n-1$. The last two examples deal with the case when $\int^{\infty} [1/a_i(s)]\, ds < \infty$ for some i and $\int^{\infty} [1/a_i(s)]\, ds = \infty$ for others, $i = 1, 2, ..., n-1$.

EXAMPLE 1. Consider the equations

$$\begin{aligned} L_{4n} x(t) + \frac{1}{t^2} x^{\alpha}[g(t)] &= \frac{1}{t^2 g^{\alpha}(t)} \sin^{\alpha}(\ln t) \\ &+ \frac{(-4)^n}{t^2} \sin(\ln t), \qquad \alpha > 0, \quad t > 0, \end{aligned} \tag{E$_1$}$$

and

$$\begin{aligned} L_{4n-2} x(t) + \frac{1}{t^2} x^{\alpha}[g(t)] &= \frac{1}{t^2 g^{\alpha}(t)} \sin^{\alpha}(\ln t) \\ &+ \frac{(-1)^n (2)^{n-1}}{t^2} \cos(\ln t), \qquad \alpha > 0, \quad t > 0, \end{aligned} \tag{E$_2$}$$

where $L_0x(t)=x(t)$, $L_kx(t)=t(L_{k-1}x(t))^{\cdot}$ for $k=1,2,...,4n$, $g(t)>0$ for $t\geqslant t_0>0$ and $g(t)\to\infty$ as $t\to\infty$.

It is easy to check that the hypotheses of Theorem 1 are satisfied for $m(t)=K$ for every $K>0$. Thus every bounded oscillatory solution $x(t)$ of (E_1) or (E_2) has the property that $L_kx(t)\to 0$ as $t\to\infty$, $k=0,1,...,m$, where $m=4n-1$ for (E_1) and $m=4n-3$ for (E_2). One such solution of (E_1) and (E_2) is $x(t)=(1/t)\sin(\ln t)$.

EXAMPLE 2. Consider the equation

$$L_{2n}x(t)+\frac{1}{t^2}x(t)=\frac{1}{t^2}\sin(\ln t)$$

$$+(-1)^n\frac{1}{t}\sin(\ln t),\qquad t>0, \tag{E_3}$$

where L_k is as in Example 1. Equation (E_3) has a solution $x(t)=\sin(\ln t)\not\to 0$ as $t\to\infty$. Only condition (6) is violated.

EXAMPLE 3. Consider the equation

$$x^{(4)}+\frac{1}{t^5}x^{1/3}=\frac{1}{t^5}[-40\cos(\ln t)-10\sin(\ln t)$$

$$+t^{-1/3}\sin^{1/3}(\ln t)],\qquad t>0. \tag{E_4}$$

One can easily verify that the hypotheses of Theorem 3 for $m(t)=K$, for every $K>0$ are satisfied and hence all bounded oscillatory solutions of (E_4) converge to zero as $t\to\infty$. We may note that one of the solutions of (E_4) is $x(t)=(1/t)\sin(\ln t)$.

EXAMPLE 4. Consider the equation

$$(t^2(t^2(t^2x^{\cdot})^{\cdot})^{\cdot})^{\cdot}+\frac{1}{t^2}x^{\alpha}[g(\mathrm{t})]=\frac{1}{t^2}\frac{1}{g^{5\alpha}(t)}\sin^{\alpha}(\ln t)+\frac{50}{t^3}\sin(\ln t)$$

$$-\frac{140}{t^3}\cos(\ln t),\qquad \alpha>0, \tag{E_5}$$

where $g(t)$ as in Example 1. One can easily check that the hypotheses of Theorem 2 and Corollary 1 are satisfied for $m(t)=K$ for every $K>0$, and that $x(t)=(1/t^5)\sin(\ln t)$ is one such solution of (E_5) satisfying the conclusion of Theorem 2.

EXAMPLE 5. Consider the equation

$$(e^{-t}(e^{t}x^{\cdot})^{\cdot})^{\cdot}+e^{-t}x(t)=2(e^{-t}+e^{-2t})\sin t. \tag{E$_6$}$$

Equation (E_5) has a solution

$$\begin{aligned}
x(t)&=e^{-t}\sin t\to 0 &&\text{as}\quad t\to\infty,\\
L_1x(t)&=\sin t-\cos t\nrightarrow 0 &&\text{as}\quad t\to\infty,\\
L_2x(t)&=e^{-t}(e^{t}x^{\cdot})^{\cdot}\to 0 &&\text{as}\quad t\to\infty.
\end{aligned}$$

Only condition (6) is violated.

EXAMPLE 6. Consider the equation

$$\begin{aligned}
&(e^{t}(e^{-t}(e^{t}x^{\cdot})^{\cdot})^{\cdot})^{\cdot}+e^{-t}x^{\alpha}[g(t)]\\
&\quad= -6e^{-t}\cos t+e^{-t-2\alpha g(t)}\sin^{\alpha} g(t), \qquad \alpha>0,
\end{aligned} \tag{E$_7$}$$

where $g(t)$ as in Example 1. All conditions of Theorem 1 are satisfied for $m(t)=K$ for every $K>0$. Equation (E_7) has an oscillatory solution $x(t)=e^{-2t}\sin t$.

Remark. We believe that the above conclusions are not deducible from other known criteria.

THEOREM 4. *Let conditions* (5) *and* (6) *hold.*
If there exists a $c_0>0$ *such that either*

$$\liminf_{t\to\infty}\int_T^t [h(s)-F(s,c_0,c_0)]\,ds>0 \tag{14}$$

or

$$\limsup_{t\to\infty}\int_T^t [h(s)+F(s,c_0,c_0)]\,ds<0 \tag{15}$$

for all large T, *then any solution* $x(t)$ *of* (1) *satisfying* (4) *is nonoscillatory.*

Proof. Let $x(t)$ be an oscillatory solution of (1) satisfying (4). Then by Theorem 1, $L_kx(t)\to 0$ as $t\to\infty$, $k=0,1,\dots,n-1$. Thus there exists $T\geqslant t_0$ such that $L_{n-1}x(T)=0$, $|x(t)|\leqslant c_0$ and $|x[g(t)]|\leqslant c_0$ for all $t\geqslant T$. Hence

$$|f(t,x(t),x[g(t)])|\leqslant F(t,c_0,c_0).$$

It follows from the above inequality that

$$h(t)-F(t, c_0, c_0) \leqslant h(t)-f(t, x(t), x[g(t)]) \leqslant h(t)+F(t, c_0, c_0)$$

for $t \geqslant T$. Thus

$$h(t)-F(t, c_0, c_0) \leqslant L_n x(t) \leqslant h(t)+F(t, c_0, c_0). \tag{16}$$

Integrating (16) form T to t we obtain

$$\int_T^t [h(s)-F(s, c_0, c_0)]\, ds \leqslant L_{n-1} x(t) \leqslant \int_T^t [h(s)+F(s, c_0, c_0)]\, ds.$$

Hence if either (14) or (15) holds, $x(t)$ cannot have arbitrarily large zeros. Which leads to a contradiction, and the proof of the theorem is complete.

COROLLARY 4. *Let conditions* (5) *hold. Assume, moreover, that either* $h(t) \geqslant 0$ *and* (14) *holds or* $h(t) \leqslant 0$ *and* (15) *holds. Then any solution* $x(t)$ *of* (1) *satisfying* (4) *is nonoscillatory.*

EXAMPLE 7. Consider the equation

$$L_{2n} x(t) + \frac{1}{t^5} x(t) = \left[1 + \frac{1}{t^4}\right] \ln t + n, \qquad t > 0, \tag{E_8}$$

where L_k is as in Example 1. All conditions of Corollary 4 are satisfied for $m(t) = t^k$, $k = 0, 1, 2, 3$. We may note that Eq. (E_8) has a nonoscillatory solution $x(t) = t \ln t$.

In our next theorems the following sublinearity type condition will be used: there exists a continuous function H: $[t_0, \infty) \to R$ such that

$$\limsup_{\substack{u \to \infty \\ u \leqslant v}} \frac{F(t, v, v)}{u} \leqslant H(t). \tag{17}$$

THEOREM 5. *In addition to conditions* (3) *and* (17), *assume that conditions* (5) *and* (6) *hold with* $m(t) = K$ *for any* $K > 0$,

$$g(t) \leqslant t \tag{18}$$

and

$$\int^\infty \frac{1}{a_1(s_1)} \int_{s_1}^\infty \cdots \int_{s_{n-2}}^\infty \frac{1}{a_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty H(s)\, ds\, ds_{n-1} \cdots ds_1 < \infty. \tag{19}$$

Then any oscillatory solution $x(t)$ *of* (1) *has the property that*

$$L_k x(t) \to 0 \quad \text{as} \quad t \to \infty, \qquad k = 0, 1, \ldots, n-1.$$

Proof. First we will show that all oscillatory solutions of (1) are bounded. Suppose that $x(t)$ is an oscillatory solution of (1) and

$$\limsup_{t \to \infty} |x(t)| = \infty.$$

Then there exists a sequence of intervals $\{(\alpha_m, \beta_m)\}_{m=1}^{\infty}$ such that $\lim_{m \to \infty} \alpha_m = \lim_{m \to \infty} \beta_m = \infty$, $x(\alpha_m) = x(\beta_m) = 0$, $|x(t)| > 0$ on (α_m, β_m) and $M_m = \max\{|x(t)| \mid t \leqslant \beta_m\} = \max\{|x(t)| \mid \alpha_m \leqslant t \leqslant \beta_m\}$ and M_m increases to ∞ as $m \to \infty$ with $M_1 \geqslant K$. Choose $P_1 < P_2 < \cdots < P_{n-1}$ as zeros of $L_k x(t)$, $k = 1, 2, \ldots, n-1$, respectively, and $\beta_m < P_{n-1}$. Now we let $t_m \in (\alpha_m, \beta_m)$ such that $|x(t_m)| = M_m$. Let $M'_m = \max\{|x(t)| \mid t \leqslant P_{n-1}\}$. Now we integrate (1) n-times to obtain

$$\begin{aligned}
M_m &= |x(t_m)| \\
&\leqslant \int_{\alpha_m}^{\beta_m} \frac{1}{a_1(s_1)} \int_{s_1}^{P_1} \frac{1}{a_2(s_2)} \int_{s_2}^{P_2} \cdots \int_{s_{n-1}}^{P_{n-1}} F(s, M'_m, M'_m)\, ds\, ds_{n-1} \cdots ds_1 \\
&\quad + \int_{\alpha_m}^{\beta_m} \frac{1}{a_1(s_1)} \int_{s_1}^{P_1} \frac{1}{a_2(s_2)} \int_{a_2}^{P_2} \cdots \int_{s_{n-1}}^{P_{n-1}} |h(s)|\, ds \cdots ds_1 \\
&\leqslant \int_{\alpha_m}^{\beta_m} \frac{1}{a_1(s_1)} \int_{s_1}^{P_{n-1}} \cdots \int_{s_{n-1}}^{P_{n-1}} F(s, M'_m, M'_m)\, ds\, ds_{n-1} \cdots ds_1 \\
&\quad + \int_{\alpha_m}^{\beta_m} \frac{1}{a_1(s_1)} \int_{s_1}^{P_{n-1}} \cdots \int_{s_{n-1}}^{P_{n-1}} |(s)|\, ds\, ds_{n-1} \cdots ds_1 \\
&\leqslant \int_{\alpha_m}^{\infty} \frac{1}{a_1(s_1)} \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} F(s, M'_m, M'_m)\, ds \cdots ds_1 \\
&\quad + \int_{\alpha_m}^{\infty} \frac{1}{a_1(s_1)} \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} |h(s)|\, ds \cdots ds_1.
\end{aligned}$$

Using condition (17) we obtain

$$\begin{aligned}
1 &\leqslant \int_{\alpha_m}^{\infty} \frac{1}{a_1(s_1)} \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} H(s)\, ds\, ds_{n-1} \cdots ds_1 \\
&\quad + \frac{1}{M_m} \int_{\alpha_m}^{\infty} \frac{1}{a_1(s_1)} \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} |h(s)|\, ds\, ds_{n-1} \cdots ds_1. \qquad (20)
\end{aligned}$$

The right hand side of (20) tends to zero as $m \to \infty$ which is a contradiction. Hence $x(t)$ is bounded and the conclusion of the theorem follows from Corollary 3.

THEOREM 6. *Let the hypotheses of Theorem 5 hold. If either* (14) *or* (15) *holds, then all solutions of* (1) *are nonoscillatory.*

Proof. Let $x(t)$ be an oscillatory solution of (1). By Theorem 5

$$L_k x(t) \to 0 \qquad \text{as} \quad t \to \infty, \quad k = 0, 1, \dots, n-1.$$

Proceeding as in the proof of Theorem 4 we obtain a contradiction.

COROLLARY 5. *Let the hypotheses of Theorem 6 hold except condition* (5). *Assume, moreover, that either $h(t) \geqslant 0$ and* (14) *holds or $h(t) \leqslant 0$ and* (15) *holds. Then all solutions of* (1) *are nonoscillatory.*

Remarks. 1. Theorems 2 and 5 in [2] are included in our Theorem 5, and the main result in [3] is included in our Corollary 5.

2. Theorem 5 is applicable to Eqs. (E_1), (E_2), (E_4), (E_5) and (E_7) with $\alpha \in (0, 1]$ and $g(t) \leqslant t$ and hence we conclude that all oscillatory solutions of these equations have the property that $L_k x(t) \to 0$ as $t \to \infty$, $k = 0, 1, \dots, n-1$.

3. SOME APPLICATIONS

We will now apply the results in the previous section to Eq. (2)

$$L_n x(t) + q(t) f(x[g(t)]) = h(t) \tag{2}$$

and obtain necessary and sufficient conditions so that all oscillatory solutions of (2) approach zero as $t \to \infty$.

We assume that

$|f(x)| \leqslant F(x)$, where F_x is positive, continuous, monotone non-decreasing, $F(xy) \leqslant F(x)\, F(y)$, $x, y > 0$, $F(0) = 0$ and

$$\int_{r_0}^{r} \frac{du}{F(u)} \to \infty \qquad \text{as} \quad r \to \infty, \qquad r \geqslant r_0 > 0, \tag{21}$$

$$0 < g(t) \leqslant t \qquad \text{and} \qquad g(t) \to \infty \qquad \text{as} \quad t \to \infty \tag{22}$$

THEOREM 7. *Suppose that condition* (11) *holds, and*

$$\int^{\infty} F(s^{n-1})\, |q(s)|\, ds < \infty, \qquad \int^{\infty} |h(s)|\, ds < \infty. \tag{23}$$

Let $x(t)$ be a solution of (2). *Then $|x[g(t)]| = O(t^{n-1})$ as $t \to \infty$.*

Proof. The proof of Theorem 7 can be modelled on that of Theorem 3.1 in [4] and hence is omitted.

THEOREM 8. *Suppose* $q(t) > 0$ *and*

$$\int^{\infty} s^{n-1}F(s^{n-1})\, q(s)\, ds < \infty \qquad \textit{for} \quad t \geqslant t_0. \tag{24}$$

Further suppose that $h(t)/F(t^{n-1})\, q(t)$ *approaches a limit as* $t \to \infty$. *Then a necessary and sufficient condition for all oscillatory solutions of* (2) *to approach zero as* $t \to \infty$ *is that*

$$\lim_{t \to \infty} \frac{h(t)}{F(t^{n-1})\, q(t)} = 0. \tag{25}$$

Proof. The proof is an adaptation of argument developed by Singh [4]. (Sufficiency). Suppose (25) holds. Then $t^{n-1}\,|h(t)| \leqslant t^{n-1}F(t^{n-1})\, q(t)$ for sufficiently large t. Since $\int^{\infty} s^{n-1}F(s^{n-1})\, q(s)\, ds < \infty$, we have $\int^{\infty} s^{n-1}\,|h(s)|\, ds < \infty$, and the conclusion follows by Theorem 3. (Necessity). Let $x(t)$ be an oscillatory solution of (2) such that $\lim_{t \to \infty} x(t) = 0$. Suppose that

$$\frac{|h(t)|}{F(t^{n-1})\, q(t)} \geqslant \gamma > 0.$$

Dividing (2) by $F(t^{n-1})\, q(t)$ and taking the limit as $t \to \infty$, we find that $L_n x(t)$ has one sign for sufficiently large t. This forces $x(t)$ to eventually assume a constant sign, which is a contradiction.

THEOREM 9. *Suppose*

$$a_i(t) \geqslant \alpha > 0, \quad i = 1, 2, ..., n-1, \quad \frac{1}{a_1(t)} = O\left(\frac{1}{t^{n-\gamma}}\right)$$
$$\textit{for some } \gamma \in [0, 1), \tag{26}$$

and

$$\int^{\infty} |q(s)|\, ds < \infty, \qquad \int^{\infty} |h(s)|\, ds < \infty, \tag{27}$$

then all oscillatory solutions of (2) *are bounded.*

Proof. The proof of Theorem 8 can be modelled on that of Lemma 2.1 in [5] and hence is omitted.

THEOREM 10. *Suppose* $q(t) > 0$ *for* $t \geqslant t_0$, *and* $\int^{\infty} s^{n-1} q(s)\, ds < \infty$. *Further, suppose that* $h(t)/q(t)$ *approaches a limit as* $t \to \infty$. *Then a necessary and sufficient condition for all oscillatory solutions of* (2) *to approach zero as* $t \to \infty$ *is that*

$$\lim_{t \to \infty} \frac{h(t)}{q(t)} = 0.$$

Proof. The proof is similar to that of Theorem 8 and is omitted.

Remark. It is easy to check that our conditions of f are weaker than those in [4, 5]. For illutration we consider the Eq. (E_4), where $f(x) = x^{1/3}$, $g(t) = t$ and $q(t) = t^{-5}$. Condition 9 in [4] fails to apply while our condition (24) does.

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